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Worldline instantons, vacuum pair production and Gutzwiller's trace formula*

Gerald V Dunne

Department of Physics, University of Connecticut, Storrs, CT 06269-3046, USA
and
Institut für Theoretische Physik, Universität Heidelberg, 69120 Heidelberg, Germany

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Abstract
In this talk, I describe a new application of the Gutzwiller trace formula formalism, to give a compact expression for the semiclassical vacuum pair production rate in quantum electrodynamics, for general inhomogeneous electromagnetic background fields. The semiclassical rate is expressed in terms of geometric properties of classical periodic phase space orbits for the associated Euclidean problem of propagation of a charged particle in the inhomogeneous background field, parametrized by the proper time.

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(Some figures in this article are in colour only in the electronic version)

Quantum vacuum fluctuations mean that the QED vacuum behaves like a polarizable medium which can be probed, for example, by boundaries (as in the Casimir effect, discussed in many talks in this workshop), or with external electromagnetic fields (the subject of this talk, among others in this workshop). An external magnetic field polarizes the vacuum by modifying the energy of the Dirac sea, while an electric field can also lead to vacuum pair production. This can be viewed as a process of ionization, in which virtual dipole pairs can be accelerated apart by the external field, becoming real asymptotic e^+e^- pairs if they gain the binding energy of $2mc^2$ from the external field (figure 1). This is a non-perturbative process, just as ionization, and the leading rate for a constant electric field was computed long ago by Heisenberg and Euler [1, 2]:

$$\frac{\operatorname{Im}\Gamma^{\mathrm{HE}}}{\mathrm{Vol}_4} \sim \frac{e^2\mathcal{E}^2}{16\pi^3}\mathrm{e}^{-\frac{\pi m^2 c^3}{e\mathcal{E}\hbar}}. \qquad (1)$$

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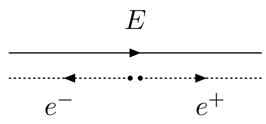

Figure 1. Pair production as separation of a virtual dipole pair.

It is instructive to compare the magnitude of the leading exponential part with the corresponding factor in the ionization rate from the ground state of hydrogen, for which $R^{\text{hydrogen}} \sim \exp\left[-\frac{2}{3}\frac{m^2 e^5}{\mathcal{E}\hbar^4}\right]$:

$$\mathrm{Im}\,\Gamma^{\mathrm{HE}} \sim \exp\left[-\frac{2}{3}\frac{m^2 e^5}{\mathcal{E}\hbar^4} \times \frac{3\pi}{2}\left(\frac{\hbar c}{e^2}\right)^3\right] \approx \exp\left[-\frac{2}{3}\frac{m^2 e^5}{\mathcal{E}\hbar^4} \times 10^7\right]. \tag{2}$$

Thus, the Heisenberg/Euler pair production rate differs from the hydrogen ionization rate by an extra factor of $\frac{3\pi}{2\alpha^3} \approx 10^7$ in the exponent!!! This fundamental difference of scales explains why the vacuum pair production phenomenon has not yet been observed directly, while atomic ionization is routinely studied with laboratory strength electric fields. Vacuum pair production is a spectacularly weak effect. Nevertheless, there are definite experimental plans to probe this effect using intense x-ray free electron lasers [3]. These lasers produce electric fields that are far from the constant field limit that was used to compute (1). This motivates a theoretical attempt to understand how the result (1) is modified by field inhomogeneities. In a very real sense, the Heisenberg/Euler result (1) is the analogue (in the context of external field probes of vacuum polarization) of Casimir's famous simple result for the Casimir energy between two ideal parallel plates. Just as much work has been done (and is the focus of much of this workshop) to extend Casimir's result to more general cases (including realistic surface properties, experimentally accessible boundary geometries and temperature dependence), so also have there been many attempts to extend the Heisenberg/Euler result to physically more realistic electric field configurations; see figures 2 and 3.

Schwinger formulated the problem in the language of renormalized quantum field theory [4]. The technical problem becomes that of computing the (non-perturbative) imaginary part of the effective action $\Gamma[A]$ in an external classical electromagnetic field represented by the gauge field $A_\mu(x)$. The vacuum pair production rate is as follows: $P_{\text{production}} = 1 - \mathrm{e}^{-2\mathrm{Im}\,\Gamma} \approx 2\mathrm{Im}\,\Gamma$. This is a very difficult problem, and surprisingly little progress has been made over the many years since Schwinger's work. The main general breakthrough is a WKB-based approach modeled on Keldysh's seminal work [5] on the theory of atomic ionization in time-dependent electric fields. The result of all this QED work is that when the background field is approximated as an electric field pointing in one fixed direction, but with amplitude that either varies with (i) time or (ii) space only along the direction of the field, then the problem can be reduced to a one-dimensional problem that can be solved (approximately) with WKB [6–9]. For example, a time-dependent electric field in the x_3 direction can be realized by an imaginary time gauge field, $A_3(x_4) = \frac{\mathcal{E}}{\omega} f(\omega x_4)$. Then, $\mathcal{E}$ characterizes the peak scale of the field amplitude, while the time inhomogeneity is characterized by the 'Keldysh adiabaticity parameter', $\gamma \equiv \frac{m\omega}{e\mathcal{E}}$. In this case, WKB modifies (1) as

$$\frac{\mathrm{Im}\,\Gamma^{\mathrm{WKB}}}{\mathrm{Im}\,\Gamma^{\mathrm{LCF}}} \approx V_3 \frac{\sqrt{2}(e\mathcal{E})^{5/2}}{32\pi^3 m\omega} \frac{\exp\left[-\frac{m^2\pi}{e\mathcal{E}}\{g(\gamma^2)-1\}\right]}{\frac{\mathrm{d}}{\mathrm{d}(\gamma^2)}(\gamma^2 g(\gamma^2))\sqrt{-\frac{\mathrm{d}^2}{\mathrm{d}(\gamma^2)^2}(\gamma^2 g(\gamma^2))}}. \tag{3}$$

Figure 2. Progression from Casimir's original configuration of perfect parallel plates, to realistic experimental configurations with different geometries and imperfect surfaces. Here vacuum polarization is probed by the boundaries.

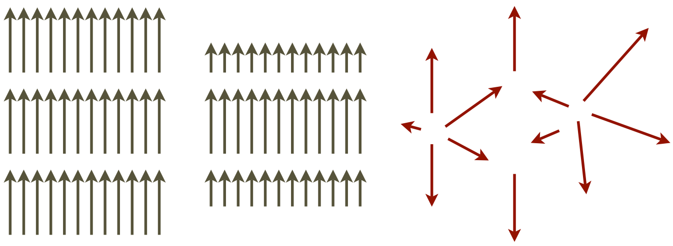

Figure 3. Progression from Heisenberg and Euler's original configuration of uniform electric field, to slightly more realistic one-dimensional inhomogeneities (center) which can be treated by WKB; but the goal in this talk is to consider the more general case of a field with multi-dimensional inhomogeneities, as shown on the right. Here vacuum polarization is probed by the applied external fields.

Here, $\mathrm{Im}\,\Gamma^{\mathrm{LCF}}$ is the *locally constant field approximation* (also called the adiabatic, or leading gradient expansion, approximation), in which one replaces the constant field in (1) with the inhomogeneous field, and integrates over spacetime. The deviation from the constant field expression is entirely characterized by the function [8] (here, $y = f(\omega x_4)/\gamma$ and f' is to be re-expressed in terms of y)

$$g(\gamma^2) \equiv \frac{2}{\pi}\int_{-1}^{1} \mathrm{d}y \frac{\sqrt{1-y^2}}{|f'|}. \tag{4}$$

The WKB approach is inherently one dimensional, and has not so far been extended to more realistic electric fields, with *multi-dimensional* inhomogeneities. This talk reports some progress in a new approach to this problem, which has recently yielded results for two- and three-dimensional spatial inhomogeneities in the electric field [10–12]. Potentially of more significance is the fact that the reformulation of the problem in the language of Gutzwiller's trace formula [13–15] suggests a well-defined approach to the computation of semiclassical pair production rates for quite general inhomogeneous electric fields [16]. The Gutzwiller trace formula has already found a wide variety of applications in theoretical and mathematical physics. In this talk, I propose a new area where its language provides insight and simplification to a difficult computational problem in relativistic quantum field theory.

The *worldline instanton* method described in this talk is based on the worldline formalism of QED [17–19], in which the effective action is expressed in terms of a *quantum mechanical* path integral in four-dimensional Euclidean space, with paths $x_\mu(\tau)$ parametrized by proper-time τ. This approach has led to many beautiful advances in our understanding of perturbative scattering amplitudes, but here we propose to use it to extract *non-perturbative* information. As is conventional, I consider scalar QED rather than spinor QED, as the computation is notationally simpler, and the leading imaginary part only differs by a spin degeneracy factor of 2. The effective action for a scalar charged particle (charge e, mass m) in a Euclidean classical gauge background $A_\mu(x)$ is the functional ($D_\mu = \partial_\mu + \mathrm{i}eA_\mu$ is the covariant derivative)

$$\begin{aligned}\Gamma[A] &= -\mathrm{tr}\ln\left(-D_\mu^2 + m^2\right)\\ &= \int_0^\infty \frac{\mathrm{d}T}{T}\,\mathrm{e}^{-m^2T}\int \mathrm{d}^4x^{(0)}\langle x^{(0)}|\mathrm{e}^{-T(-D_\mu^2)}|x^{(0)}\rangle\\ &= \int_0^\infty \frac{\mathrm{d}T}{T}\,\mathrm{e}^{-m^2T}\int \mathrm{d}^4x^{(0)}\int_{x(T)=x(0)=x^{(0)}}\mathcal{D}x\exp\left[-\int_0^T \mathrm{d}\tau\left(\frac{\dot{x}_\mu^2}{4} + eA_\mu\dot{x}_\mu\right)\right].\end{aligned} \tag{5}$$

In the last line, the trace of the associated Euclidean propagation operator has been written as a functional integral $\int\mathcal{D}x$ over all closed Euclidean spacetime paths $x_\mu(\tau)$ that are periodic (with period T) in the proper-time parameter τ [17]. We use the QED worldline path integral normalization conventions of [19].

Given this representation (5), the primary technical problem is to compute the path integral, which amounts to knowing the Euclidean propagator in the given background field. One possible approach, pioneered by Gies and Langfeld [20], with applications both to the Casimir (see the talks by Holger Gies and Klaus Klingmüller in this conference, and also [21]) and pair production problems [22], is to evaluate the Euclidean path integral numerically by Monte Carlo techniques. In a sense, one treats the path integral as 'one-dimensional lattice gauge theory', with fields $x_\mu(\tau)$ depending on the 'spacetime' coordinate τ. This approach is potentially very general, as the ensemble of loops used is independent of the form of the background field (or the geometry of the boundaries, in Casimir studies). For the pair production problem, the exponentially small imaginary part can be extracted numerically, and has been verified to work for situations with one-dimensional inhomogeneities of the electric field [22].

An alternative more analytical approach, adopted here, is to make a semiclassical (instanton) approximation to the Euclidean worldline path integral. The goal is to argue that the path integral may be dominated by a small number of special classical loops, and the quantum fluctuations about these loops. First proposed for a constant electric field [23], this idea has recently been extended to inhomogeneous background field configurations [10–12]. These special loops are solutions to the associated classical (Euclidean) equations of motion for a charged particle moving in the given (inhomogeneous) field $F_{\mu\nu}(x)$:

$$\ddot{x}_\mu = 2eF_{\mu\nu}(x)\dot{x}_\nu \qquad (\mu,\nu = 1,\ldots,4). \tag{6}$$

We call closed periodic solutions *worldline instantons*. Expanding in the fluctuations about the worldline instanton loop, $x_\mu(\tau) = x_\mu^{\mathrm{cl}}(\tau) + \eta_\mu(\tau)$, we obtain an approximate expression for the path integral:

$$\begin{aligned}\int\mathcal{D}x\,\mathrm{e}^{-S[x]} &\approx \mathrm{e}^{-S[x^{\mathrm{cl}}]}\int\mathcal{D}\eta\exp\left[-\frac{1}{2}\int_0^T \mathrm{d}\tau\,\eta_\mu\Lambda_{\mu\nu}\eta_\nu\right]\\ &= \frac{\mathrm{e}^{-S[x^{\mathrm{cl}}]}}{\sqrt{\det\Lambda}},\end{aligned} \tag{7}$$

where the fluctuation operator is

$$\Lambda_{\mu\nu} \equiv -\frac{1}{2}\delta_{\mu\nu}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + eF_{\mu\nu}(x^{\mathrm{cl}})\frac{\mathrm{d}}{\mathrm{d}\tau} + e\partial_\mu F_{\rho\nu}(x^{\mathrm{cl}})\dot{x}_\rho^{\mathrm{cl}}. \tag{8}$$

In [10] it was shown that the worldline instanton exponential factor $\mathrm{e}^{-S[x^{\mathrm{cl}}]}$ agrees precisely with the WKB exponential factor in the case of one-dimensional inhomogeneities.

It is important to note that the fluctuation operator Λ is an *ordinary* differential operator. Thus, one can use the Gel'fand–Yaglom approach (which only applies to ordinary differential operators) to compute the determinant in an efficient manner [25–28]. The Gel'fand–Yaglom result says that the determinant can be computed *without computing the eigenvalues of* Λ. (This is crucial to the method, because this step should be done for any T, as we need to integrate over T in (5).) One solves instead the 'Jacobi equation', $\Lambda_{\mu\rho}\eta_\rho(\tau) = 0$, with *initial value boundary conditions*:

$$\Lambda_{\mu\rho}\eta_\rho(\tau) = 0, \qquad \eta_\mu^{(\nu)}(0) = 0, \qquad \dot{\eta}_\mu^{(\nu)}(0) = \delta_{\mu\nu} \qquad (\mu, \nu = 1, \ldots, 4). \tag{9}$$

Then the Gel'fand–Yaglom result states that the infinite-dimensional functional determinant of Λ can be expressed in terms of a finite-dimensional determinant constructed from the values of the four independent solutions $\eta^{(\nu)}$, evaluated at $\tau = T$:

$$\det(\Lambda) = \det\left[\eta_\mu^{(\nu)}(T)\right]. \tag{10}$$

Knowing (possibly only numerically) the worldline instanton solution $x^{\mathrm{cl}}(\tau)$, it is straightforward to implement numerically this computation of the determinant of the fluctuation operator.

In fact, in [11] it was shown that for one-dimensional inhomogeneities the determinant of the fluctuation operator can be computed analytically. For such fields, the fluctuation operator is effectively 2×2 (rather than 4×4), involving just $(x_3(\tau), x_4(\tau))$, and remarkably one is able to find *analytically* all four independent solutions to the Jacobi equation. Thus one can also find analytically the two solutions satisfying the initial value boundary conditions (9), and hence the determinant. One finds [11]

$$\det(\Lambda) = \left(2m\dot{x}_4^{\mathrm{cl}}(T)\int_0^T \frac{\mathrm{d}\tau}{\left[\dot{x}_4^{\mathrm{cl}}(\tau)\right]^2}\right)^2. \tag{11}$$

Further, after doing the T integral by steepest descents, one reproduces exactly the WKB result (3). This is a nice non-trivial test of the worldline instanton method, but the really interesting thing is that (unlike WKB) the method can be extended to more general multi-dimensional inhomogeneities. In [12], the semiclassical pair production rate was computed for spatially inhomogeneous static electric fields, whose magnitude and direction varied in two and three dimensions. Technically, there are several steps to the computation. (i) Find periodic solutions (the worldline instantons) to the Euclidean classical equations of motion (6), such that the conserved quantity $\frac{1}{4}\dot{x}_\mu^2$ takes the value m^2. This latter condition follows from the T integral, as is explained below. (ii) The dominant exponential factor in the pair production rate is $\mathrm{e}^{-S[x^{\mathrm{cl}}]}$, involving the classical action evaluated on the worldline instanton path. (iii) The prefactor coming from the semiclassical approximation to the quantum mechanical path integral can be evaluated using the Gel'fand–Yaglom result (10). (iv) The T integration fixes the conserved quantity $\frac{1}{4}\dot{x}_\mu^2$ to take the value m^2, and also produces a prefactor from Laplace's method. (v) Possible residual dependence on the spacetime location of the loop must be integrated over. In a Gaussian approximation, this also leads to another prefactor.

Note that in general there are three separate (but below we shall see that they are in fact related!) prefactor contributions to the final answer: one coming from each integration in (5). This procedure is reminiscent of that used to derive the Gutzwiller trace formula [13–15] for the trace of Green's function in non-relativistic quantum mechanics:

$$\mathrm{tr}\, G(E) = \int \mathrm{d}^3x \int_0^\infty \mathrm{d}t\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}Et}\langle x|\mathrm{e}^{\frac{\mathrm{i}}{\hbar}Ht}|x\rangle \tag{12}$$

$$= \sum_{\text{orbits p}} T_p \frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_p(E) - \mathrm{i}\frac{\pi}{2}m_p}}{\sqrt{\det(1 - \mathbf{J}_p)}}. \tag{13}$$

Here, T_p is the period of the pth orbit having energy E, $S_p(E)$ is its action, $\mathbf{J}_p$ is the associated monodromy matrix (see below) and m_p are the Maslov indices. Before proceeding to the corresponding worldline instanton expression, we briefly sketch the strategy of the derivation of the Gutzwiller trace formula (an excellent introduction for physicists is in [15]). The first step is to make a semiclassical approximation for the propagation kernel in (12):

$$\langle x|\mathrm{e}^{\frac{\mathrm{i}}{\hbar}Ht}|x'\rangle \approx \sqrt{\det\left|\frac{\partial^2 R}{\partial x\,\partial x'}\right|}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}R(x,x';t)}. \tag{14}$$

Here, $R(x, x'; t)$ is Hamilton's principal function for the path connecting the points x and x', in time t. The prefactor is the Van Vleck determinant factor, given by variations with respect to the path's endpoints. It is tempting at this stage to simplify this prefactor, by writing it in the Gel'fand–Yaglom form [28], but in fact it is better to leave it as it stands for now. The next step is to evaluate the t integral in (12) by stationary phase. Here we note that the exponent is $(Et + R(x, x'; t))$, whose stationary point condition, $\frac{\partial R}{\partial t} = -E$, fixes t to be the period such that the closed classical path has energy equal to E. Then this stationary point condition implements the Legendre transform from $R(x, x'; t)$ to the action $S(x, x'; E)$ associated with the closed path of energy E. Furthermore, the prefactor from the t integral contributes a factor $1\big/\sqrt{-\frac{\partial^2 R}{\partial t^2}} = \sqrt{\frac{\partial^2 S}{\partial E^2}}$, which follows from the other Legendre transform relation $\frac{\partial S}{\partial E} = t$. The final step is to integrate over the marked point $x^{(0)}$ on the closed worldline instanton loop. Evaluating this integral by a third and final steepest descents approximation forces the closed loop is to be periodic. The $x^{(0)}$ integral naturally splits into an integral along the loop, and transverse to the loop. The integral along the loop produces a factor of the period t, due to translation invariance. The transverse integrations give yet another determinant prefactor $1\big/\sqrt{\det\left|\frac{\partial^2 S}{\partial x_\perp \partial x'_\perp}\right|}$. The remarkable result of Gutzwiller is that these three different-looking prefactors (coming successively from semiclassical approximations to the quantum mechanical path integral, the t integral and finally the x trace) all combine into a single determinant prefactor that has a simple geometrical interpretation in phase space:

$$\int \mathrm{d}x \left[\frac{\sqrt{\frac{\partial^2 R}{\partial x \partial x'}}\sqrt{\frac{\partial^2 S}{\partial E^2}}}{\sqrt{\frac{\partial^2 S}{\partial x_\perp \partial x'_\perp}}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(x,x';E)}\right]_{x=x'} = t\,\frac{\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S[E]}}{\sqrt{\det(\mathbf{1}-\mathbf{J})}}, \tag{15}$$

where $\mathbf{J}$ is the monodromy matrix of the periodic orbit having energy E and period t. For a precise definition of $\mathbf{J}$, see [13–15], including the Maslov indices, which we have ignored here for the sake of simplicity. For our purposes, it is sufficient to observe that the determinant is a classical invariant of the closed orbit, which characterizes the behavior of small deviations from the periodic orbit in phase space.

Now comes the important point. These steps we have just sketched for the derivation of Gutzwiller's trace formula are closely analogous to the steps required in evaluating the worldline path integral expression (5) for the effective action $\Gamma[A]$. Analogous to (13), we seek the following representation of the imaginary part of the effective action:

$$\begin{aligned}\mathrm{Im}\,\Gamma &= \mathrm{Im}\int \mathrm{d}^4x \int_0^\infty \frac{\mathrm{d}T}{T}\,\mathrm{e}^{-m^2T}\langle x|\mathrm{e}^{-(-D_\mu^2)T}|x\rangle \\ &= \sum_{\text{orbits p}} \frac{\mathrm{e}^{-S_p(E=m^2)}}{\sqrt{\det(1-\mathbf{J}_p)}}.\end{aligned} \tag{16}$$

It is a Euclidean path integral, in four-dimensional space rather than three, and there is an extra factor of $1/T$ because $\Gamma[A]$ is a log determinant. Nevertheless, the ideas follow through and at the end we can combine the three prefactors found by following the worldline instanton strategy enumerated above, to form one simple prefactor expressed in terms of the monodromy matrix for the periodic orbit, now viewed in phase space. We first approximate the quantum mechanical path integral as

$$K(x,x';T) := \langle x|\mathrm{e}^{-T(-D_\mu^2)}|x'\rangle \approx \frac{1}{(2\pi)^2}\sqrt{\left|\det\left(\frac{\partial^2 R}{\partial x\partial x'}\right)\right|}\,\mathrm{e}^{-R(x,x';T)}, \tag{17}$$

where $R(x,x';T)$ is the Hamilton principal function for the classical trajectory from x to x' in four-dimensional Euclidean space, in the proper-time interval T. This classical trajectory is obtained by solving the Euclidean classical equations of motion, the worldline instanton equations (6). To evaluate the trace in (5) we will need the diagonal propagation kernel $K(x^{(0)},x^{(0)};T)$, but for now we consider the point-split propagation from x to x'. The classical equations of motion (6) are those for a charged particle moving in an inhomogeneous electromagnetic field $F_{\mu\nu}(x)$, so the 'energy' is conserved on a classical trajectory: $E = \frac{1}{4}\dot{x}_\mu^2 = \text{constant}$. Next, the T integral is evaluated by steepest descents. The critical point of the exponential factor arises when $\frac{\partial R}{\partial T} = -m^2$. This has a natural classical interpretation in terms of the Legendre transformation between the Hamilton principal function $R(x,x';T)$ (expressed in terms of the total time elapsed along the trajectory) and the action $S(x,x';E)$ (expressed in terms of the constant energy of the trajectory): $R(x,x';T) = S(x,x';E) - ET$. It follows that $\frac{\partial R}{\partial T} = -E$ and $\frac{\partial S}{\partial E} = T$. Thus, the critical point T_c of the T integral occurs when $E = m^2$, so that

$$\int_0^\infty \frac{\mathrm{d}T}{T}\,\mathrm{e}^{-m^2 T}K(x,x';T) \approx \frac{1}{(2\pi)^2 T_c}\sqrt{\left|\det\left(\frac{\partial^2 R}{\partial x\partial x'}\right)\right|_{T_c}}\sqrt{\frac{2\pi}{\left|\frac{\partial^2 R}{\partial T^2}\right|_{T_c}}}\,\mathrm{e}^{-S(x,x';m^2)}. \tag{18}$$

The two prefactor contributions combine in a simple way if we consider the coordinates $x_\parallel^{(0)}$ along the classical trajectory and $x_\perp^{(0)}$ transverse to the trajectory. Then it follows from classical mechanics that [13–15]

$$\left.\frac{\det\left(\frac{\partial^2 R}{\partial x\partial x'}\right)}{\frac{\partial^2 R}{\partial T^2}}\right|_{T_c} = \frac{1}{\dot{x}_\parallel \dot{x}'_\parallel}\det\left(\frac{\partial^2 S(x,x';m^2)}{\partial x_\perp \partial x'_\perp}\right). \tag{19}$$

The final step is the coincident limit $x \to x' = x^{(0)}$ and trace over $x^{(0)}$. This trace is also done by steepest descents and forces the closed loop to be periodic [13–15]. From (19), the integration over $x_\parallel^{(0)}$ yields a factor $\int \mathrm{d}x_\parallel^{(0)}\big/\dot{x}_\parallel^{(0)} = T_c/2$ (translation invariance along the periodic orbit), while the $x_\perp^{(0)}$ integral produces another determinant factor. This determinant factor combines with the remaining transversal determinant factor in (19) to give $\det(\mathbf{1}-\mathbf{J})$:

$$\frac{\det\left(\frac{\partial^2 S(x,x';m^2)}{\partial x_\perp\partial x_\perp} + \frac{\partial^2 S(x,x';m^2)}{\partial x'_\perp\partial x_\perp} + \frac{\partial^2 S(x,x';m^2)}{\partial x_\perp\partial x'_\perp} + \frac{\partial^2 S(x,x';m^2)}{\partial x'_\perp\partial x'_\perp}\right)}{\det\left(\frac{\partial^2 S(x,x';m^2)}{\partial x_\perp\partial x'_\perp}\right)}$$
$$= \det\left(\frac{\partial(p_\perp - p'_\perp, x_\perp - x'_\perp)}{\partial(x'_\perp, p'_\perp)}\right) =: \det(\mathbf{1}-\mathbf{J}). \tag{20}$$

Here all determinants are evaluated at vanishing transverse displacements. Here, $\mathbf{J}$ is the *monodromy matrix* for a six-dimensional surface of section in phase space transverse to the periodic phase space orbit with constant energy $E = m^2$. Consider an initial transverse displacement $(\delta x'_\perp, \delta p'_\perp)^T$ from a point on the closed orbit in phase space, and evolve for time T, and the final displacement from the orbit is related to the initial one by the monodromy matrix: $(\delta x''_\perp, \delta p''_\perp)^T = \mathbf{J}(\delta x'_\perp, \delta p'_\perp)^T$. Putting all these parts together, and collecting phases carefully, one obtains a compact final expression:

$$\mathrm{Im}\,\Gamma \approx \frac{\mathrm{e}^{-S(E=m^2)}}{\sqrt{\det(\mathbf{1}-\mathbf{J})}}. \tag{21}$$

The principal advantage of expressing the computation in this language of the Gutzwiller trace formula is that the total prefactor is encapsulated in a *single determinant*, which moreover has a natural mathematical and geometrical meaning in the Euclidean phase space. In previous works [7, 8, 9, 11], the various prefactor contributions have been evaluated separately, and then combined at the end. Thus, the computational strategy is as follows.

(i) Solve the classical equations of motion in four-dimensional Euclidean space to find all closed periodic trajectories of energy $E = m^2$: the 'worldline instanton(s)'.
(ii) Evaluate the classical action $S(E = m^2)$ on these trajectories. Then the leading exponential contribution to $\mathrm{Im}\,\Gamma$ is $\mathrm{e}^{-S(m^2)}$.
(iii) Compute the prefactor from the monodromy matrix $\mathbf{J}$ for the dominant trajectory(ies).

The only concrete comparison we can make is to compute $\mathrm{Im}\,\Gamma$ for the case of a one-dimensional inhomogeneity, which can be computed in several other ways [7–9, 11]. Consider, for example, the case of a time-dependent electric field directed in the x_3 direction. We can choose a Euclidean gauge field $A_3(x_4) = \frac{\mathcal{E}}{\omega} f(\omega x_4)$, where $\mathcal{E}$ characterizes the overall magnitude of the associated electric field, ω characterizes the scale of the time dependence and $f(\omega x_4)$ is some smooth function. For example, for a constant electric field $\mathcal{E}(t) = \mathcal{E}$, we have $f(x) = x$; for a sinusoidal electric field $\mathcal{E}(t) = \mathcal{E}\cos(\omega t)$, we have $f(x) = \sinh(x)$; and for a single-pulse electric field $\mathcal{E}(t) = \mathcal{E}\mathrm{sech}^2(\omega t)$, we have $f(x) = \tan(x)$. Then the classical action on a periodic trajectory of energy E can be written (here, $y := \frac{e\mathcal{E}}{\omega\sqrt{E}} f(x)$) as

$$S(E) = \oint \mathrm{d}x_4 \sqrt{E - \left(\frac{e\mathcal{E}}{\omega} f(\omega x_4)\right)^2} = \frac{2E}{e\mathcal{E}} \int_{-1}^{1} \mathrm{d}y \frac{\sqrt{1-y^2}}{[f'(z)]_{z=z(y)}}. \tag{22}$$

This is precisely the exponent appearing in the standard result for the pair production rate [7–9, 11]. To evaluate the prefactor, we can choose x_4 as $x_\parallel$. Then the transverse x_3 direction is in fact an invariant 'flat' direction, so we do not need to perform the transverse integration. This illustrates the important point that (21) must be interpreted appropriately when there are physical zero modes. Thus, we go back to (18) and observe that $\frac{\partial^2 R}{\partial T^2} = -1\big/\frac{\partial^2 S}{\partial E^2}$. Furthermore, the other determinant factor in (18) is easily computed (see [11]) using the Gel'fand–Yaglom formula:

$$\det\left(\frac{\partial^2 R}{\partial x \partial x'}\right)\bigg|_{x=x'} = \frac{m^4}{16E^3T^2}\frac{1}{\dot{x}_4^2\left(\frac{\partial^2 S}{\partial E^2}\right)^2}. \tag{23}$$

Thus, relative to the constant spatial volume V_3,

$$\frac{\mathrm{Im}\,\Gamma}{V_3} \approx \frac{\sqrt{2\pi}}{2(4\pi)^2 m}\left[\frac{\mathrm{e}^{-S(E)}}{\frac{\partial S}{\partial E}\sqrt{\frac{\partial^2 S}{\partial E^2}}}\right]_{E=m^2}. \tag{24}$$

Note that (24) agrees precisely with the conventional WKB result in (3), with the added semiclassical interpretation of the various terms.

I conclude with some comments and open problems. (i) Finding closed periodic orbits to (6) is an interesting and non-trivial problem, and recasting the question in phase space proves helpful. (ii) If the physical electric field is too localized in space, then we know physically that the pair production rate vanishes (since the virtual vacuum dipole pairs cannot gain enough energy from the field to become real electron–positron pairs). In simple cases this corresponds to the non-existence of periodic classical Euclidean trajectories [10, 11]. It would be interesting if this were more generally true: that the mere existence of such worldline instanton loops might be used as an *indicator* of pair production. (iii) The phases arising from the steepest descent integrals combine to give $\mathrm{Im}\,\Gamma$ in the cases where the electric field is a function of either t or $\vec{x}$ [11], but the general mixed case needs further analysis. (iv) If the gauge field corresponding to the external field can be put into the nonlinear gauge, where $A_\mu^2(x) = \text{constant}\,(\equiv E)$, then we can solve the simpler first-order equations $\dot{x}_\mu = -2eA_\mu(x)$, as was observed long ago by Nambu [29]. (v) It would be interesting to extend our method to inhomogeneous non-Abelian fields, for which little is known beyond simple quasi-Abelian cases. This suggests studying the Wong equations [30] describing the classical motion of a color-charged particle in a non-Abelian background, which may have physical implications for the color glass condensate [31, 32].

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